Final exam Robust Control

31 October 2023, 15:00–17:00

The exam consists of 4 exercises. Please write clearly and provide clear explanations to your answers. The maximum possible number of points is 40.

1 $(2+2+3+3=10$ points) **Stabilizability and stabilization**

Consider the system Σ:

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t).$$

Here, u is the control input, x is the state and y is the measured output.

(a) Define the notions of stabilizability of the pair (A, B) and detectability of (C, A).

(b) State the Hautus lemmas (eigenvalue tests) for stabilizability and detectability.

(c) Assume that (A, B) is stabilizable and (C, A) is detectable. Provide a recipe for finding a stabilizing controller

$$\begin{aligned} \dot{w}(t) &= Kw(t) + Ly(t) \\ u(t) &= Mw(t) + Ny(t) \end{aligned}$$

for the system Σ.

(d) Assume that (K, L, M, N) is a stabilizing controller for Σ, equivalently, the matrix

$$\begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}$$

is Hurwitz. Prove that (A, B) is stabilizable and (C, A) is detectable.

2 $(3+3+4=10$ points) **H_2 optimal control**

Consider the system Σ of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t),\ z(t) = Cx(t) + Du(t).$$

Here, u is the control input, d the disturbance input, and z the performance output.

(a) Give the precise definition of the H_2 optimal control problem by state feedback.

(b) Assume that (A, B) is stabilizable and the problem is in standard form. Explain how to compute the optimal performance and (if it exists) an optimal state feedback.

(c) Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d,\ z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Compute the optimal performance. Does there exist an optimal state feedback? If so, compute one.

3 (3 + 5 = 8 points) H_∞ norm

Consider the system

$$\dot{x} = Ax + Bu,\ y = Cx + Du,$$

with transfer matrix $G(s) = C(sI - A)^{-1}B + D$. Assume that A is Hurwitz.

(a) Give the definition of the H_∞ norm $||G||_\infty$.

(b) Assume that there exists a $Y \geq 0$ such that

$$\begin{bmatrix} A^\top Y + YA + C^\top C & YB + C^\top D \\ B^\top Y + D^\top C & D^\top D - \gamma^2 I \end{bmatrix} \leq 0.$$

Prove that $||G||_\infty \leq \gamma$.
Hint: Work out the expression $\frac{d}{dt}(x(t)^\top Y x(t)) + ||y(t)||^2 - \gamma^2 ||u(t)||^2$.

4 (3 + 3 + 3 + 3 = 12 points) Robust stabilization

Consider the nominal system Σ_n:

$$\dot{x}(t) = Ax(t) + Bu(t),\ \ y(t) = Cx(t),$$

and the generalized plant Σ:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t),\ \ y(t) = Cx(t) + D_{12}d(t),\ \ z(t) = C_2x(t) + D_{21}u(t) + D_{22}d(t).$$

(a) Give a precise definition of the problem of robust stabilization.

(b) Give a precise formulation of the small gain theorem.

(c) Explain how the solution to the H_∞ optimal control problem can be used to solve the robust stabilization problem.

(d) Explain what is meant by additive perturbations. What are the matrices E, D_{12}, C_2, D_{21} and D_{22} is this case?

End of exam, cheers!